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Symmetry Characterization of Pimenov's Spacetime: A Reformulation of Causality Axioms

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In modern physics, many theories are formulated in terms of symmetries; therefore, if we want to incorporate, e.g., space-time physics into modern physics, it is desirable to reformulate space-time physical theories in terms of symmetries. In this paper, we provide such a reformulation for an axiomatic theory of kinematic causality.

1. FORMULATION OF THE PROBLEM

In modern physics (starting from the discovery of quarks in the early 1960s) symmetries are the language in which many physical ideas are formulated. This is especially true in quantum field theory. Therefore, if we want to incorporate other physical formalisms into modern physics, it is desirable to translate these other formalisms into the language of symmetries. This is especially important for space-time geometry, which is not well accommodated with quantum field theory.

Part of space-time physics is described in terms of differential equations (Einstein's, Brans–Dicke theory, etc). This part can be reformulated in terms of symmetries of these equations in the sense that we can find enough symmetries to uniquely reconstruct the equations; see, e.g., Finkelstein and Kreinovich (1985) and Finkelstein *et al.* (1986). However, differential equations have limitations (Misner *et al.*, 1973): they cannot describe nonsmoothness due to singularities and nonsmoothness due to quantum fluctuations of space-time geometry. To describe nonsmooth space-times, a more general formalism was proposed in Busemann (1967), Kronheimer and Penrose (1967), and Pimenov (1970). In this formalism, the basic notion is *kinematic*

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causality, an ordering relation $a < b$ that, crudely speaking, means that a and b can be connected by a timelike curve. This formalism has been used for physical applications (see, e.g., Kreinovich, 1974, 1979; Gromov *et al.*, 1976). The main goal of this paper is to *reformulate the axioms of kinematic causality in terms of symmetries.*

2. MAIN IDEA

When we think of a symmetric space-time, the most natural example is a *homogeneous* space-time, i.e., a space-time in which for every two points a and b, there exists a symmetry that transforms a into b. Plane (Minkowski) space-time is homogeneous even in a stronger sense: namely, if we have two pairs of points (a, b) and (a', b') that are *causally similar* in the sense that either $a < b$ and $a' < b'$, or $a > b$ and $a' > b'$, etc., then there exists a symmetry that transforms a into a' and b into b'. In other words, if 2-element subsets $\{a, b\}$ and $\{a', b'\}$ are isomorphic, then the isomorphism f: $\{a, b\} \to \{a', b'\}$ can be extended.

In general, space-times are not necessarily homogeneous, so we cannot formulate that every isomorphism can be extended. However, as we will see, some extension properties can be formulated for generic space-time models, and these extension properties turn out to be exactly equivalent to the axioms of kinematic causality formulated in Busemann (1967), Kronheimer and Penrose (1967), and Pimenov (1970).

3. AXIOMS OF KINEMATIC CAUSALITY

Definition 1 (Pimenov, 1970). An ordered set $(M, <)$ is called a *kinematic space* if it satisfies the following axioms:

K_1 $\forall a\, \exists b, c(b < a < c)$.
K_2 $\forall a, b(a < b \to \exists c(a < c < b))$.
K_3 $\forall a, b, c(a < b, c \to \exists d(a < d < b, c))$.
K_4 $\forall a, b, c(a > b, c \to \exists d(a > d > b, c))$.

Comment. Usually, the additional condition is imposed that a space-time should be *directed* in the following sense:

Definition 2. A kinematic space is called *directed* if the following two conditions are true:

D_1 $\forall a, b\, \exists c(a, b < c)$.
D_2 $\forall a, b\, \exists c(a, b > c)$.

4. REFORMULATION IN TERMS OF SYMMETRY

Before we reformulate these conditions in terms of symmetries, let us recall the definitions of isomorphism and homomorphism of ordered sets.

Definition 3:

- A mapping $f: M \to M'$ between ordered sets is called a *homomorphism* if for every $a, b \in M$ for which $a < b$ we have $f(a) < f(b)$.
- A mapping $f: M \to M'$ is called an *isomorphism* if it is 1–1, and both f and the inverse mapping are homomorphisms.

Definition 4. Let $n > 0$ be an integer. We say that an ordered set $(M, <)$ is *n-structurally homogeneous* if for every isomorphism $f: S \to S'$ between two subsets $S, S' \subseteq M$ with n or fewer elements, and for every $m \notin S$, there exists an element $m' \notin S'$ such that a mapping f, extended to $S \cup \{m\}$ by setting $f'(m) = m'$, is a homomorphism $f': S \cup \{m\} \to S' \cup \{m'\}$.

It turns out that 3-structural homogeneity is (almost) equivalent to kinematic space axioms, almost in the sense that there are (nonphysical) degenerate cases when they are not equivalent:

Definition 5. We say that elements a, b of an ordered set M are *incompatible* (and denote it $a \parallel b$) if $a \neq b$, $a \not< b$, and $b \not< a$.

Definition 6. An ordered set $(M, <)$ is called *nondegenerate* if the following two conditions are satisfied:

- $\exists a, b, c(a < c \;\&\; b < c \;\&\; a \parallel b)$.
- $\exists a, b, c(a > c \;\&\; b > c \;\&\; a \parallel b)$.

Comment. In the real world, both in the past and in the future there are pairs of events that are not causally connected (truly simultaneous). If a space-time were not nondegenerate in the sense of this definition, this would mean that either in the past or in the future, the space-time was Newtonian (linearly ordered). Both cases contradict our current understanding of space-time.

Theorem. A nondegenerate ordered set M is 3-structurally homogeneous iff M is a directed kinematic space.

Comment. If we exclude nonphysical degenerated ordered sets, we can say that we have reformulated the axioms of directed kinematic space in terms of symmetries: namely, it is equivalent to 3-structural homogeneity.

5. PROOF

1. Let us first assume that $(M, <)$ is a nondegenerate 3-structurally homogeneous ordered set. We want to prove that M is a directed kinematic space.

1.1. Let us first prove K_1. Let $a \in M$. We want to prove that there exists a c for which $a < c$. Since M is nondegenerate, there exist $p, q, r \in M$ for which $p < q$, $p < r$, and $q \parallel r$. So, due to 3-structural homogeneity, for $S = \{p\}$, $S' = \{a\}$, $f(p) = a$, and $m = q$, there exist an m' and a homomorphism f': $\{p, q\} \rightarrow \{a, m'\}$ for which $f'(q) = m'$. Since f' is a homomorphism and $p < q$, we have $a < m'$. Hence, we can take $c = m'$. Similarly, we can prove that there exists a b for which $b < a$.

1.2. Let us now prove K_2. Let $a < b$, and let us prove that there exists a c for which $a < c < b$. Take an arbitrary point $p \in M$. Due to K_1, for $a = p$, there exists $q > p$; applying the same formula K_1 to $a = q$, we conclude that there exists $r > q$. From $r > q > p$, we conclude that $r > p$. Let us now apply 3-structural homogeneity to $S = \{p, r\}$, $S' = \{a, b\}$, $f(p) = a$, $f(r) = b$, and $m = q$. Then we conclude that for some $m' \in M$, the mapping f': $\{p, q, r\} \rightarrow \{a, m', b\}$ for which $f'(p) = a$, $f'(r) = b$, and $f'(q) = m'$ is a homomorphism. From $p < q < r$, we conclude that $a < m' < b$. So, K_2 is proven for $c = m'$.

1.3. Let us now prove K_3. Let $a < b$ and $a < c$. We want to show that there exists a d for which $a < d$, $d < b$, and $d < c$. To prove it, we will consider all possible cases of a relationship between b and c:

1. $b = c$.
2. $b < c$.
3. $b > c$.
4. $b \parallel c$.

1.3.1. For $b = c$, K_3 directly follows from K_2.

1.3.2. For $b < c$, from K_2, it follows that there exists a d for which $a < d < b$. Then, from $d < b < c$, we conclude that $d < c$.

1.3.3. Similar to 1.3.2.

1.3.4. Let us now consider the case when $a < b$, $a < c$, and $b \parallel c$. Due to K_1, there exists a $b' < a$. Let us apply 3-structural homogeneity to $S = \{b', b, c\}$, $S' = \{a, b, c\}$, $f(b') = a$, $f(b) = b$, $f(c) = c$, and $m = a$. Then, there exists an m' such that the mapping f extended to f' by setting $f'(a) = m'$ is a homomorphism. Therefore, from $b' < a < b, c$, we conclude that $f'(b') < f'(a) < f'(b), f'(c)$, i.e., that $a < m' < b, c$. Therefore, we can take $d = m'$.

1.4. K_4 is proven similarly to K_3.

1.5. We have proven that M is a kinematic space. Let us now prove that M is directed. We will prove D_1 (D_2 can be proven similarly). So, $a, b \in$

M, and we want to find c for which $a < c$ and $b < c$. Let us consider all possible cases of a relationship between a and b:

1. $a = b$.
2. $a < b$.
3. $a > b$.
4. $a \parallel b$.

1.5.1. If $a = b$, then the existence of the desired c follows from K_1.

1.5.2. Let $a < b$. Then, applying K_1 to b, we get a c for which $c > b$. Hence, $c > b > a$, and $c > a$.

1.5.3. Similarly to 1.5.2.

1.5.4. Finally, let us consider the case when $a \parallel b$. Since M is nondegenerate, there exist p, q, and r such that $q < p$, $r < p$, and $q \parallel r$. Let us now apply 3-structural homogeneity to $S = \{q, r\}$, $S' = \{a, b\}$, $f(q) = a, f(r) = b$, $m = p$. Then, we get an $m' = f'(p)$ for which from $p > q, r$ we conclude that $m' = f'(p) > f'(q) = a$ and similarly, that $m' > b$. So, we can take $c = m'$.

2. To complete the proof of the theorem, we must show that if M is a nondegenerate kinematic space, then M is 3-structurally homogeneous. In other words, we want to prove that for every isomorphism $f: S \to S'$ between two subsets $S, S' \subseteq M$ with $s \leq 3$ elements, and for every $m \notin S$, there exists an element $m' \notin S'$ such that a mapping f, extended to $S \cup \{m\}$ by setting $f'(m) = m'$, is a homomorphism f': $S \cup \{m\} \to S' \cup \{m'\}$. Let $S = \{s_1, \ldots, s_s\}$. We will consider all possible relations between m and s_i:

2.1. If $m \parallel s_i$ for all i, then we can take $m' > s'_1$, which exists due to K_1. In this case, f' is a homomorphism (but not an *iso*morphism).

2.2. If $m > s_i$ or $m \parallel s_i$ for all i, then we will take the following m':

2.2.1. If $s = 1$, we take $m' > s'_1$, which exists due to K_1.

2.2.2. If $s = 2$, then we take $m' > s'_1, s'_2$, which exists due to D_1.

2.2.3. If $s = 3$, then, due to D_1, there exists p for which $p > s'_1$ and $p > s'_2$. Applying D_1 to p and s'_3, we get m' for which $m' > p > s'_1$, $m' > s'_2$, and $m' > s'_3$.

2.3. If $m < s_i$ or $m \parallel s_i$ for all i, then, similarly, we can take m' for which $m' < s'_1$ for all i.

2.4. Let us now consider the remaining case, in which for some i, $m > s_i$, and for some j, $m < s_j$. The total number of such i and of such j cannot exceed the number of elements in S and is therefore ≤ 3. Hence, we have the following possibilities:

2.4.1. If there is only one such i and only one such j, then, clearly $s_i < s_j$. Since f is an isomorphism, we have $s'_i < s'_j$. We can take m' for which $s'_i < m' < s'_j$ (the existence of such m' is guaranteed by K_2).

2.4.2. If there is one i for which $s_i < m$ and two j for which $m < s_j$, i.e., if $s_i < m < s_j, s_k$, then $s_i < s_j, s_k$. Hence, due to isomorphism between S and S', $s'_i < s'_j, s'_k$, and we can take m' for which $s'_i < m' < s'_j, s'_k$, whose existence is guaranteed by K_3.

2.4.3. Similarly, if there are two i's and one j, we can use K_4.

In all cases, 3-structural homogeneity is proven. QED

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